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A PROPOSED STAGGERED-GRID SYSTEM FOR NUMERICAL INTEGRATION OF DYNAMIC EQUATIONS

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ABSTRACT

A system is proposed for grid allocation and differencing of apparently general applicability to purely marching-type systems of equations of fluid dynamics. The method is based on casting of the equations into the conservation form, which then permits use of a staggered space-time grid system with interpolations required only in certain linear terms. The method is illustrated by application to two systems of equations, on one of which numerical experiments have been successfully performed. Advantages and drawbacks of the method are described in comparison to other currently used grid systems, and the possibility and desirability of parametric simulation of turbulent eddy exchange processes are discussed.

1. INTRODUCTION

The recent development of methods of numerical integration of the equations of fluid dynamics in a more-or-less primitive form, i.e., of first order, except for diffusion terms, is focusing attention upon new aspects of numerical analysis. During the early development of geostrophic or balanced models which involved a vorticity equation, perhaps the most pressing practical problems were to solve rapidly and accurately certain rather complex elliptic second order partial differential equations. In dealing with undifferentiated systems one finds that the basic equations are much simpler to apply but must, for the sake of computational stability, be applied at very short time intervals compared to the time scales of the significant meteorological phenomena. This is because the normal Courant-Friedrichs-Lewy stability criterion requires essentially that motions or waves allowed in the system not be able to travel from one grid point to the next in one time step, and the undifferentiated systems usually allow faster-moving waves than those of meteorological interest. It is also noted that the boundary conditions and various consistency requirements, if not more complex, are more critical when a single computation run may include several thousand time steps. Under these circumstances it is desirable to examine the time-space grid data allocation and differencing scheme with a view toward eliminating, if feasible, some of the redundant time resolution and thus save time and computation expense. It also appears that when integrating a set of non-linear equations over a moderate number of time steps, say several hundred, a certain form of instability arises, which is related to spatial truncation error in the non-linear terms. This non-linear instability, discussed by Phillips [7], seems to occur much more rapidly in primitive equations models than in those using a vorticity equation, though it is uncertain whether this is due to differences in the physical or mathematical behavior of the systems. At any rate there appear to be several possible devices for elimination of this instability, which will be mentioned later.

As a method of eliminating some of the computational redundancy, Eliassen [3] proposed a method of handling a two-level primitive equations model, in which variables are staggered in space and time in a somewhat complex manner. The principle on which the system was based was that the linear terms of the equations would be available at the correct grid points with minimal truncation error, but that certain non-linear terms would be formed from products of interpolated values. The reason for doing this, presumably, was that the linear terms of the motion equations, the pressure gradient and Coriolis terms, are generally an order of magnitude larger than the others and errors in these terms might be thought to be most damaging. On the other hand the advective terms may contribute as much, and in many cases much more, toward significant rates of change of quantities than the combination of the nearly balanced linear terms. As long as truncation error does not disturb the existing quasi-balance between the linear terms which is most pronounced for the larger scales of motion, there seems no reason to require interpolation to be done only on the advective terms. It will be shown that the truncation error of interpolation in the proposed system is, at most, of the same order as that arising from the finite difference pressure gradient term, but that the latter is evaluated with less truncation error in the Eliassen grid than in that proposed.

Hinkelmann [4] and Smagorinsky [13] have successfully applied hydrostatically filtered equations of two-dimensional flow without the use of time or space staggering of the variables. In both of these models the external gravitational motions were filtered out by the vanishing boundary conditions on dp/dt, and it was therefore necessary to solve a Poisson equation at each time step. Both used the marching equations in the conservation form, and it was found that non-linear computational instability ensued, in the absence of specific damping terms, within two or three days of commencement of integration. Phillips [8] has recently developed a barotropic divergent (free surface) primitive equations model, in which the Eliassen grid system has been applied, with central time and space differencing, and also with a one-sided space difference scheme [9] oriented according to the wind direction and related to that originally proposed by Courant, Isaacson, and Rees [1]. The latter method introduces some computational damping and seems effectively to prevent development of non-linear instability.

2. PRINCIPLES OF PROPOSED SYSTEM AND APPLICATION

A method will now be presented for stepwise integration of initial-value boundary-value problems using non-linear hydrodynamic equations. The method appears to be generally applicable to meteorological problems when the system of equations is of an explicit marching type, that is when there are no physical approximations or constraints which involve the solution of an elliptic equation at each time step. It is evidently not particularly suitable for application to, for example, the barotropic vorticity equation, or to primitive equations models such as those of Smagorinsky and Hinkelmann in which external gravity waves are filtered out, because in these systems further interpolations would be necessary. A principal feature of the method is the casting of the equations into the conservation form, the form in which all advective terms appear as flux divergences. When this is done it becomes very natural to set up a central time and space differencing system on a space-time staggered grid, in which interpolations may be required only in certain linear terms. The method of interpolation is largely optional, but time interpolation is generally most accurate. If frictional terms are present (and it is believed that they generally should be) these are most naturally computed by forward differencing, which then simultaneously preserves linear computational stability in these terms. The method will be illustrated by application to a free surface model similar to that of Phillips.

If the atmosphere is considered as a homogeneous incompressible fluid with a free upper surface, whose height is ϕ/g, the two-dimensional, hydrostatically filtered equations of motion may be written, in Cartesian coordinates, as

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}-f\epsilon_{ij3}u_j+\frac{\partial \phi}{\partial x_i}=\frac{1}{\phi}\frac{\partial \tau_{ij}}{\partial x_j} \tag{1}$$

where ϵ_{ijk} is the permutation tensor, equal to plus or minus unity for $i, j, k=1, 2, 3$ or $2, 1, 3$ respectively.

Upon application of the viscosity hypothesis, but without assuming constant viscosity, the frictional stress may be written as

$$\tau_{ij}=\phi K\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2\delta_{ij}}{\delta_{ii}}\frac{\partial u_k}{\partial x_k}\right). \tag{2}$$

The repeated index implies summation, so that δ_{ii} equals two in this case. If we were considering a physical system whose characteristic Reynolds number was small, for example a "dishpan" experiment, we could reasonably assume ϕK constant and the stress terms would reduce to the common Laplacian form. For the atmospheric system we want the stress tensor to describe the transfers of energy, momentum, etc., performed by turbulent eddy motions, the scales of which range between that of the smallest explicitly described motions and that of molecular dissipation. Smagorinsky [14] proposes that for fully turbulent flow K be made proportional to the deformation, i.e.

$$K=(kl)^2\left|\frac{\tau_{ij}}{\phi K}\right| \tag{3}$$

where l is the scale of the smallest resolvable motions (the grid scale in a finite difference formulation) and k is a universal constant of order unity. From consideration of the similarity of the above expression to that frequently applied in turbulent boundary layer theory we consider k to be analogous to the Kármán constant. The appearance of a particular length scale in the definition is a reflection of the essential correspondence of the frictional stress to the Reynolds stress, also defined in terms of the same length scale.

The form proposed by Smagorinsky is similar to one applied by von Neumann and Richtmyer [10] in their "pseudo-viscosity" method for representing shock waves. It has been introduced here because of its seeming importance in elimination of the non-linear computational instability which is otherwise permitted by the proposed grid differencing scheme (and many others in current use). The physical significance and justification of these terms will be shown by Smagorinsky and are not discussed further here.

The continuity equation is a prediction equation for the height of the free surface, written as:

$$\frac{\partial\phi}{\partial t}+\frac{\partial(\phi u_j)}{\partial x_j}=0. \tag{4}$$

If (4) is multiplied by u_i and combined with (1), a momentum equation is obtained, that is

$$\frac{\partial(\phi u_i)}{\partial t}+\frac{\partial}{\partial x_j}(\phi u_i u_j)-f\epsilon_{ij3}\phi u_j+\frac{\partial(\phi^2/2)}{\partial x_i}=\frac{\partial\tau_{ij}}{\partial x_j}. \tag{5}$$

We now assume that motions within a rectangular area of dimensions L_i are cyclically symmetric in all directions, so that the boundary conditions may be written, in terms of an arbitrary dependent variable χ, as

$$\chi(x_1, x_2, t)=\chi(x_1\pm L_1, x_2, t)=\chi(x_1, x_2\pm L_2, t). \tag{6}$$

Equations (4), (5), and (6) are a complete set of prediction equations for the dependent variables, ϕu_i, ϕ, written in the conservation form.

We divide the $L_1\times L_2$ area into pq grid squares, each of area Δ^2, where p and q are both even numbers. Letting l and m represent the index numbers of grid intersections, and n a time index, we define

$$\chi_{l,m}^{(n)}=\chi(l\Delta, m\Delta, n\Delta t)=\chi_{l\pm p,m}^{(n)}=\chi_{l,m\pm q}^{(n)} \tag{7}$$

the latter two equalities proceeding directly from (6).

Let us now specify all the dependent variables at the even intersections ($l+m$ even) for even n and also all of them at the odd intersections for odd n. We then may approximate (4) and (5) by finite difference equations as follows:

$$\phi_{l,m}^{(n+1)}=\phi_{l,m}^{(n-1)}-\frac{\Delta t}{\Delta}[(\phi u_1)_{l+1,m}-(\phi u_1)_{l-1,m}+(\phi u_2)_{l,m+1}-(\phi u_2)_{l,m-1}]^{(n)} \tag{8}$$

$$(\phi u_1)_{l,m}^{(n+1)}=(\phi u_1)_{l,m}^{(n-1)}-\frac{\Delta t}{\Delta}[(\phi u_1^2+\phi^2/2-\tau_{11})_{l+1,m}-(\phi u_1^2+\phi^2/2-\tau_{11})_{l-1,m}+(\phi u_1u_2-\tau_{12})_{l,m+1}-(\phi u_1u_2-\tau_{12})_{l,m-1}]^{(n)}+2f\Delta t\,\overline{(\phi u_2)}_{l,m}^{(n)} \tag{9}$$

$$(\phi u_2)_{l,m}^{(n+1)}=(\phi u_2)_{l,m}^{(n-1)}-\frac{\Delta t}{\Delta}[(\phi u_1u_2-\tau_{21})_{l+1,m}-(\phi u_1u_2-\tau_{21})_{l-1,m}+(\phi u_2^2+\phi^2/2-\tau_{22})_{l,m+1}-(\phi u_2^2+\phi^2/2-\tau_{22})_{l,m-1}]^{(n)}-2f\Delta t\,\overline{(\phi u_1)}_{l,m}^{(n)}. \tag{10}$$

The stresses and the viscosity coefficient K may also be approximated, from (2) and (3) as:

$$(\tau_{11})_{l,m}^{(n)}=-(\tau_{22})_{l,m}^{(n)}=(\phi_{l,m}^{(n)}K_{l,m}^{(n-1)}/\Delta)[(u_1)_{l+1,m}-(u_1)_{l-1,m}-(u_2)_{l,m+1}+(u_2)_{l,m-1}]^{(n-1)} \tag{11}$$

$$(\tau_{12})_{l,m}^{(n)}=(\tau_{21})_{l,m}^{(n)}=(\phi_{l,m}^{(n)}K_{l,m}^{(n-1)}/\Delta)[(u_1)_{l,m+1}-(u_1)_{l,m-1}+(u_2)_{l+1,m}-(u_2)_{l-1,m}]^{(n-1)} \tag{12}$$

$$K_{l,m}=k^2\Delta\{[(u_1)_{l+1,m}-(u_1)_{l-1,m}-(u_2)_{l,m+1}+(u_2)_{l,m-1}]^2+[(u_1)_{l,m+1}-(u_1)_{l,m-1}+(u_2)_{l+1,m}-(u_2)_{l-1,m}]^2\}^{1/2}. \tag{13}$$

The bar over the Coriolis terms in (9) and (10) indicates that some form of interpolation or replacement must be done, since the quantities involved are not defined at the proper time-space grid points.

Four possible methods of interpolation are suggested. Spatial interpolation would involve replacement of the barred quantities in (9) and (10) as follows, according to either a 2-point or 4-point formula.

$$\left.\begin{array}{l}\overline{(\phi u_1)}_{l,m}^{(n)} \text{ is replaced by } 1/2\,[(\phi u_1)_{l,m+1}+(\phi u_1)_{l,m-1}]^{(n)}\\ \text{and}\\ \overline{(\phi u_2)}_{l,m}^{(n)} \text{ is replaced by } 1/2\,[(\phi u_2)_{l+1,m}+(\phi u_2)_{l-1,m}]^{(n)}\end{array}\right\} \tag{14}$$

or

$$\overline{(\phi u_i)}_{l,m}^{(n)} \text{ is replaced by } 1/4\,[(\phi u_i)_{l+1,m}+(\phi u_i)_{l-1,m}+(\phi u_i)_{l,m+1}+(\phi u_i)_{l,m-1}]^{(n)}. \tag{15}$$

Other methods of spatial interpolation would involve somewhat larger truncation errors. For time interpolation one would use similar replacements as follows:

$$\overline{(\phi u_i)}_{l,m}^{(n)} \text{ is replaced by } 1/2\,[(\phi u_i)_{l,m}^{(n+1)}+(\phi u_i)_{l,m}^{(n-1)}] \tag{16}$$

where (9) and (10) would then have to be linearly combined to eliminate the implicit terms. The resulting system will be shown to have negligible truncation error in the Coriolis terms, as well as a slightly less restrictive computational stability criterion. A simpler method involving somewhat larger but still acceptable truncation errors is a forward-backward scheme similar to that introduced by Courant, Friedrichs, and Lewy [2] for a linear wave problem. In this case the barred quantities in (9) and (10) are replaced as follows:

$$\left.\begin{array}{l}\overline{(\phi u_1)}_{l,m}^{(n)} \text{ is replaced by } (\phi u_1)_{l,m}^{(n-1)}\\ \overline{(\phi u_2)}_{l,m}^{(n)} \text{ is replaced by } (\phi u_2)_{l,m}^{(n+1)}.\end{array}\right\} \tag{17}$$

It may be noted that the frictional terms in (11), (12), and (13) involve velocity derivatives at time index $(n-1)$ weighted by ϕ values at time index (n). This inconsistency (if such it is) seems from experience to be totally unimportant. It has, in fact, been convenient to compute similar frictional terms over considerably longer time steps than the remainder of the calculations, and little error is so introduced. It is well known, however, that straightforward central differencing for the diffusion equations may lead to linear computational instability (see, e.g., Richtmyer [10]).

With regard to initial conditions, it may be desired to begin computations with values of variables all pertaining to the same time, rather than in the staggered grid. In this case it is satisfactory to make an initial half-time step, and similarly to interpolate between time steps in order to obtain a simultaneous display at a later time.

3. TRUNCATION ERROR AND COMPUTATIONAL STABILITY

We shall not attempt to give a complete discussion of the computational stability and truncation error of finite difference formulations of (1)–(4). We have, however, determined the stability properties of linearized versions of several finite difference equivalents of the frictionless system and first approximations to the truncation errors of various terms in the motion equations. This material is assembled in tables 1 and 2. The expressions given for truncation error are the lowest order terms of the Taylor series, and are presented in this form for comparative purposes rather than for any attempt at numerical evaluation. The grid formulations used include (a) the normal advective form, based on equation (1) with central differencing and no interpolations; (b) the proposed staggered grid system with 2-point and 4-point space interpolations, time interpolation, and forward-backward treatment of the Coriolis terms; and (c) the Eliassen grid as used by Phillips [8]. Figures 1 and 2 illustrate the essential difference between these grid systems. In the normal advective scheme all variables are defined at all intersections of figure 1, while in the proposed staggered system all quantities are defined for even time-steps at grid points marked "E" and for odd time at points "O". In the Eliassen scheme the geopotential, ϕ, is defined on grid "A" of figure 2 for even time and grid "D" for odd time, u_1 is defined on grid "B" for odd time and grid "C" for even, and u_2 on grid "C" for odd time and grid "B" for even. The grid interval shown on figure 2 was altered to $(1/\sqrt{2})$ of that defined by Phillips in order that the truncation errors for the various systems be comparable. Thus in all cases the total number of points describing a given field, combining odd and even time steps, is given by the ratio of the total area to Δ^2.

Table 1 shows that the non-viscous linear computational stability characteristics are similar for all the grid-differencing systems considered. The differences lie in the Coriolis term, which is ordinarily two to three orders of magnitude smaller than that arising from the external gravity wave propagation. It may be easily demonstrated that the friction terms, computed in the form described, exert an effect on the stability criterion of at most the same order of magnitude as the advection.

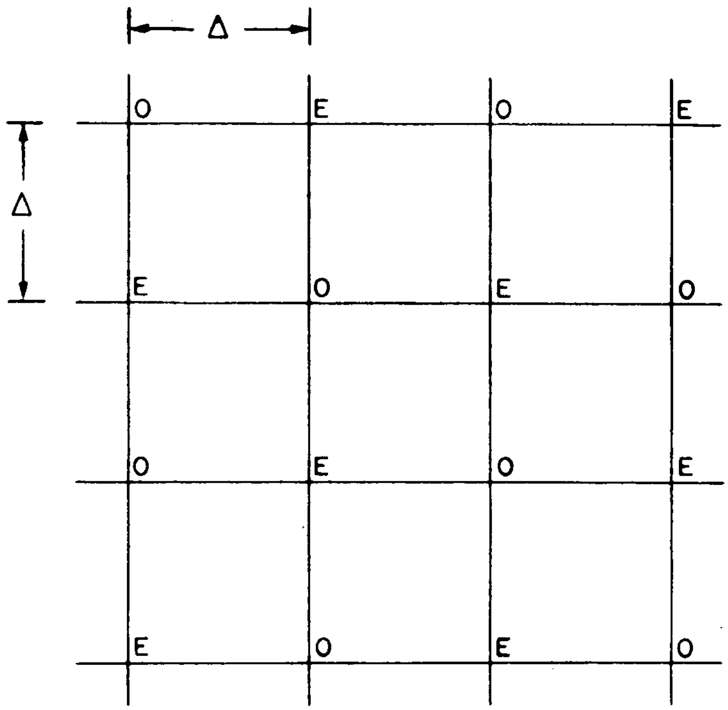

FIGURE 1.—Grid system for normal advective scheme, in which all variables are defined at all intersections; and for proposed staggered system, in which all quantities are defined for even time-steps at points E and for odd time at points O.

Upon careful examination of table 2 we find that truncation errors also do not present any clear superiority of one system over another. Truncation errors in the geostrophic terms are certainly minimized in the Eliassen scheme, because of the smaller differencing interval in the pressure term. Nevertheless the unavoidable error of first-order differencing of the pressure term is of the same order as that of spatial interpolation of the balancing Coriolis term, so that introduction of the latter breeds no

Table 1.—*Linear computational stability criteria*

Normal advective	$\frac{\Delta}{\Delta t} > \lvert u_i\rvert + \sqrt{2\phi + (f\Delta)^2}$
Space-interpolated staggered	$\frac{\Delta}{\Delta t} > \lvert u_i\rvert + \sqrt{2\phi + (f\Delta)^2}$
Time-interpolated staggered	$\frac{\Delta}{\Delta t} > \lvert u_i\rvert + \sqrt{2\phi}$
Forward-backward staggered	$\frac{\Delta}{\Delta t} > \lvert u_i\rvert + \sqrt{2\phi + (f\Delta)^2}$
Eliassen	$\frac{\Delta}{\Delta t} > \lvert u_i\rvert + \sqrt{2\phi + (f\Delta)^2/2}$

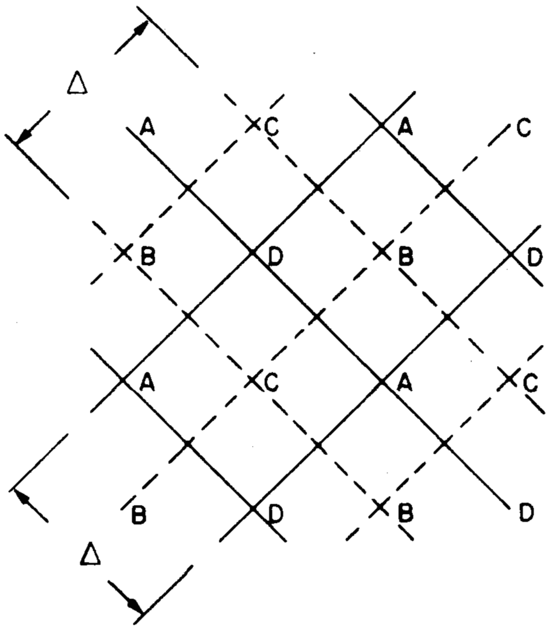

FIGURE 2.—The Eliassen scheme in which the geopotential ϕ is defined on grid A for even time and on grid D for odd time, u_1 is defined on grid B for odd time and on grid C for even time, and u_2 is defined on grid C for odd time and on grid B for even time.

new sources of error. Proper comparison of the various methods of interpolating the Coriolis term is dependent upon specification of characteristic time and space scales of motion. For typical meteorological motions (small gravity-wave amplitude) the time interpolation has the smallest error, only a little larger than that of the time derivative term. The error of the forward-backward scheme is considerably larger and may be nearly as large as that of the space-interpolation method, depending on the spatial smoothness of the fields. Comparison of the truncation errors associated with the non-linear terms is more difficult. It appears, however, that terms of all systems are of roughly the same size except the Eliassen component along the wind, which is about four times larger.

All results available to the author indicate that all the systems considered above exhibit non-linear computational instability unless their short-wave components are effectively damped. Our understanding of this phenomenon is, however, severely restricted by the present lack of general methods of its analysis. Phillips' [7] method may be used to prove instability, in some cases, but it can never prove stability. Trial-and-error methods are rather inefficient, with the large number of possible methods to choose among. Shuman [11] has, however, recently performed one-dimensional numerical experiments, the results of which raise the possibility that certain reasonable

TABLE 2.—*Truncation errors of terms in equations of motion*

1. Pressure derivative term; e.g. $\frac{\partial\phi}{\partial x_1}$	
Normal advective	$\frac{\Delta^2}{6}\frac{\partial^3\phi}{\partial x_1^3}$
Staggered	$\frac{1}{\phi}\frac{\Delta^2}{12}\frac{\partial^3\phi^2}{\partial x_1^3}$
Eliassen	$\frac{\Delta^2}{12}\frac{\partial^3\phi}{\partial x_1^3}$
2. Coriolis term; e.g. fu_2	
Normal advective	None
2-point space-interpolated staggered	$\frac{f}{\phi}\frac{\Delta^2}{8}\frac{\partial^2(\phi u_2)}{\partial x_1^2}$
4-point space-interpolated staggered	$\frac{f}{\phi}\frac{\Delta^2}{16}\nabla^2(\phi u_2)$
Time-interpolated staggered	$\frac{f}{\phi}\frac{(\Delta t)^2}{2}\frac{\partial^2(\phi u_2)}{\partial t^2}$
Forward-backward staggered	$\frac{f}{\phi}(\Delta t)\frac{\partial(\phi u_2)}{\partial t}$
Eliassen	None
3. Time derivative, e.g. $\frac{\partial u_1}{\partial t}$	
Normal advective	$\frac{(\Delta t)^2}{6}\frac{\partial^3 u_1}{\partial t^3}$
Staggered	$\frac{1}{\phi}\frac{(\Delta t)^2}{6}\frac{\partial^3(\phi u_1)}{\partial t^3}$
Eliassen	$\frac{(\Delta t)^2}{6}\frac{\partial^3 u_1}{\partial t^3}$
4. Advection term along wind component; e.g. $u_1\frac{\partial u_1}{\partial x_1}$	
Normal advective	$\frac{\Delta^2}{6}u_1\frac{\partial^3 u_1}{\partial x_1^3}$
Staggered	$\frac{1}{\phi}\frac{\Delta^2}{6}\frac{\partial^3(\phi u_1^2)}{\partial x_1^3}$
Eliassen	$\frac{\Delta^2}{6}\left(\frac{\partial^2}{\partial x_1^2}+3\frac{\partial^2}{\partial x_2^2}\right)\frac{\partial u_1^2}{\partial x_1}$
5. Advective term across wind component; e.g. $u_2\frac{\partial u_1}{\partial x_2}$	
Normal advective	$\frac{\Delta^2}{6}u_2\frac{\partial^3 u_1}{\partial x_2^3}$
Staggered	$\frac{1}{\phi}\frac{\Delta^2}{6}\frac{\partial^3(\phi u_1 u_2)}{\partial x_2^3}$
Eliassen	$\frac{\Delta^2}{12}u_2\left(3\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)\frac{\partial u_1}{\partial x_2}$

difference schemes may be stable without damping. If these results can be shown to apply to a more general model we may have a close counterpart to the spectral method which in some cases may conserve energy identically.

The eddy-viscosity term proposed by Smagorinsky obviously exerts a damping which is highly selective in scale. For a given amplitude of disturbance kinetic energy and a given grid interval the value of K is proportional to the finite difference approximation of the deformation amplitude. When central differencing is used each contribution to this amplitude will be proportional to sin $(2\pi\Delta/\lambda)$, where λ is the wavelength of a motion component. Thus damping is maximized for wavelengths at and near 4Δ and actually vanishes at $\lambda=2\Delta$. Since components with wavelengths between 2Δ and 4Δ are most directly involved in non-linear instability it is therefore suggested that the viscosity term prevents the instability indirectly. That is, it reduces the amplitudes of the intermediate scale (stable) components which would otherwise generate small-scale (unstable) components. If this is the case we may have reasonable confidence that the important effects of the analytic viscosity terms (2), (3) are fairly well approximated by their finite difference formulation.

Other methods that have been used to prevent non-linear instability do more or less violence to the physics. Probably the worst method is to use a constant viscosity, or its equivalent in a difference system or smoothing filter. Experiments have shown that in order to be effective the viscosity must be large enough so that the *greatest* characteristic grid Reynolds number of the field is of order unity, where by contrast we note that Smagorinsky's eddy viscosity is of such a form that the grid Reynolds number is *everywhere* of order unity. In order to simulate motions of a turbulent fluid one would then have to carry millions of grid points, otherwise the system would be essentially laminar and the large-scale components severely oversmoothed. Lax [6] suggested a space-time staggered grid for integration of equations in the conservation form. Forward-time differencing was used and variables at the previous time were space-averaged before differencing. This process effectively introduces a very large constant computational viscosity, essentially equal to $\Delta^2/4\ \Delta t$, and the grid Reynolds number is of order $u/\sqrt{\phi}\ll 1$. On somewhat the opposite extreme of selectivity is the method used by Phillips [7] in his general circulation calculations, which consisted of Fourier-analyzing the motion field and removing all components with wavelengths less than 4Δ. Somewhere between these extremes lies the one-sided difference scheme first suggested by Courant, Isaacson, and Rees [1], in which the advective form of the vector equation of motion is used and spatial derivatives are taken along the characteristics (streamlines) upward of the central point. Variants of this method have been used by Kasahara [5] and Phillips [9]. Motions are damped, but rather selectively, and the general results appear to be very similar to those obtained using Smagorinsky's eddy friction form, although the physical significance is unclear.

4. OTHER COMPUTATIONAL FEATURES

An advantage of the proposed system, in comparison with, for example, the Eliassen grid, is the vanishing truncation error of volume integrals (numerical sums) of the prediction variables. Smagorinsky [12] has shown that this is assured by application of certain auxiliary boundary conditions (mainly symmetry conditions) and these in turn prevent development of another type of computational instability. The importance of this feature probably depends somewhat on the use to which the particular prediction system is put. For general circulation studies, where integrated momentum budgets, heat budgets, etc. are of fundamental interest, it seems very desirable to know that, for example, the mean geopotential is only a function of the boundary conditions. For operational forecasting this may be of less direct interest, though perhaps comforting knowledge. In any case it has proven to be valuable for "debugging" machine program logic, for determination of machine errors, and for evaluation of round-off errors.

The proposed system has certain other practical advantages for application to a high-speed electronic computer, not all of which are shared by other systems. First, only one set of variables need be stored for each point, in distinction to the two or three required for a non-staggered grid using forward-differenced viscous terms. Second, there are just two kinds of points, rather than four as in the Eliassen system, which makes the logic somewhat simpler. On the other hand, application of any staggered grid system adds definite logical complications to a machine program, thus increasing the programming and check-out time and, to a small extent, the time spent by the machine in logical testing.

5. APPLICATION TO OTHER MODELS

The proposed method was originally developed for, and applied to, integration of a set of two-dimensional (x_1, x_3) equations used for the simulation of dry convective motions. The system used was the following:

$$\frac{\partial(\rho u_i)}{\partial t}+\frac{\partial}{\partial x_j}(\rho u_i u_j)+\frac{\partial p}{\partial x_i}+g\rho\delta_{i3}=\frac{\partial \tau_{ij}}{\partial x_j}$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_j}(\rho u_j)=0$$

$$\frac{\partial(\rho\theta)}{\partial t}+\frac{\partial}{\partial x_j}(\rho u_j\theta)=\frac{\partial H_j}{\partial x_j}$$

where pressure is obtained from the equation of state in the form

$$\frac{p}{p_0}=\left(\frac{R\rho\theta}{p_0}\right)^{\kappa}$$

and the friction-diffusion terms were of a similar form, though more complex, than those described above.

The grid scheme outlined above is clearly applicable to this set, where the linear density term in the vertical momentum equation must be interpolated. In this case it is possible to interpolate density either spatially or timewise. Integrations have been successfully performed using two-point (vertical) and four-point spatial interpolation and linear time interpolation, the results of which will be reported elsewhere.

If the above set of equations were modified to exclude sound wave propagation, by elimination of the time derivative of density and formation of a vorticity equation, it would then not be practical to apply the proposed system. On the other hand, it can evidently be applied to a multi-level hydrostatically filtered baroclinic model, provided that the vertically integrated divergence is not constrained. The applicability of the system depends not on the scale or complexity of the model but only on the absence of a constraining elliptic differential equation.

6. SUMMARY

It has been demonstrated theoretically and, in one particular case, practically, that the staggered grid system here proposed can be advantageously applied in the integration of purely marching type hydrodynamic-thermodynamic models. The method is of a similar type to Eliassen's but essentially opposite in its basic principles, in that here the linear terms may be interpolated, spatially or perhaps preferably in time, while the advective terms are applied with centered differencing in the conservation form. Physical considerations suggest the abstraction of energy from grid-scale motions by parametric simulation of eddy exchange processes, and Smagorinsky's suggested method for doing this seems to be sufficient for maintenance of computational stability. The essential features of the system have been described within the framework of a rotating incompressible barotropic fluid model with a free surface, while its practical application has been to a somewhat more complex model of a compressible fluid with thermal convection.

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